't Hooft GR exercise 10.2 pg 45

WTS: $\partial_\mu (\sqrt{-g} F^{\mu\nu}) = - \sqrt{-g} J^\nu$ is

equivalent to $D_\mu F^{\mu\nu} = - J_\nu$

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = \frac{1}{2} \frac{1}{\sqrt{-g}} (-\partial_\mu g) F^{\mu\nu} + \sqrt{-g}\, \partial_\mu F^{\mu\nu}$$

$$= -\frac{1}{2} \frac{\partial_\mu g}{\sqrt{-g}} F^{\mu\nu} + \sqrt{-g}\, \partial_\mu F^{\mu\nu} = -\sqrt{-g} J^\nu$$

$\Rightarrow$ WTS

$$\frac{1}{2} \frac{\partial_\mu g}{g} F^{\mu\nu} + \partial_\mu F^{\mu\nu} = - J^\nu$$

is equivalent to

$$D_\mu F^{\mu\nu} = - J^\nu ,$$

in other words, $D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \frac{1}{2} \frac{\partial_\mu g}{g} F^{\mu\nu}$

Where as usual, $g \equiv \det (g_{\mu\nu})$.

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\lambda\mu} F^{\lambda\nu} + \Gamma^{\nu}_{\lambda\mu} F^{\mu\lambda}$$

$\Gamma^{\nu}_{\lambda\mu}$ symmetric in λ, μ,

$F^{\mu\lambda} = \partial^\mu F^\lambda - \partial^\lambda F^\mu$ antisymmetric

$$= \partial_\mu F^{\mu\nu} + \Gamma^{\mu}_{\lambda\mu} F^{\lambda\nu}$$

$$= \partial_\mu F^{\mu\nu} + \Gamma^{\alpha}_{\mu\alpha} F^{\mu\nu}$$

$$\Gamma^{\alpha}_{\mu\alpha} = \frac{1}{2} g^{\alpha\sigma} \left[g_{\sigma\mu,\alpha} + g_{\sigma\alpha,\mu} - g_{\mu\alpha,\sigma} \right]$$

$\Uparrow$

symmetric in α, σ

antisymmetric in α, σ

$$= \frac{1}{2} g^{\alpha\sigma} g_{\sigma\alpha,\mu}$$

$$\Rightarrow D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \frac{1}{2} g^{\alpha\sigma} g_{\sigma\alpha,\mu}$$

$\Uparrow$

$\frac{1}{2} \frac{\partial_\mu g}{g}$?

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